A Multiharmonic Rejection Filter Designed by an Exact Method

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Summary—The exact method for designing band-stop filters in transmission lines is here adapted to design a four-element filter that is perfectly matched at a fundamental frequency and has infinite attenuation (theoretically) at the second, third, and fourth harmonics. The form of the filter is suitable for construction in TEM-mode strip transmission lines. How to obtain other combinations of three infinite rejection frequencies is also shown. Each filter is derived from a Cauer-type prototype network obtained from published tables of element values. The computed response of a test design is seen to be a precise mapping of the response of the prototype.

Introduction

A FILTER composed of three open-circuited stubs separated by two quarter-wavelength transmission-line sections can be designed to have a perfect match at one frequency and infinite attenuation (theoretically, if the circuit were dissipationless) at any chosen (harmonic or other) frequency [1]. The design method is facilitated by available tables of element values for a low-pass filter prototype [2]–[4]. By using exact design techniques and formulas for band-stop filters in transmission line [5]–[7], subject to certain added constraints on the choice of a prototype, the engineer can readily complete the design. The low-pass prototype alluded to above is a simple LC ladder network consisting of three reactive elements ($n=3$).

The purpose of this paper is to extend this design method to the case of a harmonic rejection filter having infinite attenuation for three clustered[1] harmonic frequencies. With this slightly more complex method there is still zero attenuation of the fundamental frequency. Instead of having pass bands between the three specified harmonic frequencies, the filter attenuation falls to a minimum value fixed by the choice of the prototype element values. The prototype is a (low-pass) Cauer-type network ($n=3$ or 4) having one finite frequency of infinite attenuation. Exact design formulas are used [6], and published tables of element values for these filters [8] are employed, as before.

By means of an exact mapping procedure [5]–[7] the frequency of perfect match ω_v' shown in the attenuation curve of the low-pass prototype (Fig. 1), is made to map into the fundamental frequency ω_v of the transmission line filter (Fig. 2). At the same time, the frequency of infinite attentuation ω_2' is mapped into, say, the second and fourth harmonics of the transmission line filter, $2\omega_v$ and $4\omega_v$ respectively, while infinite frequency in the frequency plane of the prototype is mapped into the third harmonic $3\omega_v$. Other combinations are possible, as is explained below. Note that for this case, $3\omega_v$ is also labelled ω_0 which is the design frequency for which the line sections are a quarter-wavelength long. In this procedure the second (or middle) infinite rejection frequency is always the design frequency ω_0, from which the other two infinite rejection frequencies are equi-spaced.

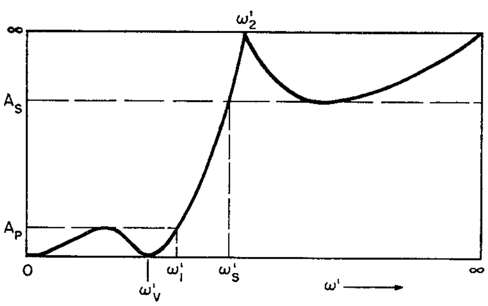

Fig. 1—Attenuation of a low-pass filter ($n=4$) with Chebyshev response, vs frequency normalized to the fundamental ω_v.

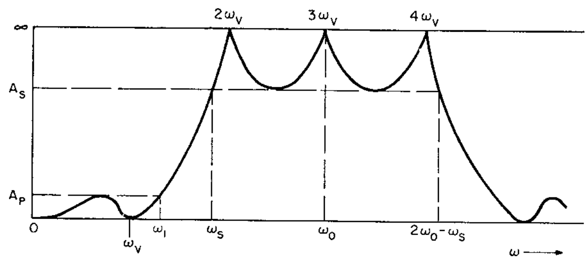

Fig. 2—Attentuation of a transmission-line filter that is matched at the fundamental frequency, has infinite attenuation of the second, third, and fourth harmonics, and which is derived from the prototype by an exact method.

As with all transmission-line filters designed by this method, the response shape (as in Fig. 2, for example) is periodic in ω with an interval $2\omega_0$, and each period has a symmetry axis at odd multiples of ω_0.

The design method consists in first determining the normalized value of ω_2' (Fig. 1) needed for the mapping process, which is fully defined in Table I. This frequency, ω_2', and the specified maximum pass-band attenuation A_p uniquely determine the prototype ele-

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[1] The lower and higher harmonic frequencies must be equi-spaced from the middle frequency.

TABLE I

DEFINITION OF PARAMETERS AND MAPPING FUNCTION

1. *Definition of Parameters*
   - ω' = Prototype frequency
   - ω = Transmission line filter frequency
   - $\Lambda = a\omega_2'$
   - $a = \cot(\pi/2\ \omega_2/\omega_0)$
   - ω_2' = frequency of infinite attenuation of the prototype filter
   - ω_2 = lowest of three harmonic frequencies to be rejected
   - ω_0 = design frequency at which all line sections are $\lambda_0/4$ long; also the middle harmonic frequency
2. *Mapping Function*
   - $\omega' = \Lambda \tan(\pi/2\ \omega/\omega_0)$
3. *Corresponding Frequencies*

ω' plane	ω-plane	
0	$m\omega_0$ (m is always even)	Center frequency of all pass bands
ω_v'	$m\omega_0 \pm \omega_v$	Frequency of perfect match (fundamental)
ω_1'	$m\omega_0 \pm \omega_1$	Upper edge of pass band
ω_s'	$m\omega_0 \pm \omega_s$	Beginning of stopband
ω_2'	$m\omega_0 \pm \omega_2$	Infinite rejection frequencies
∞	$n\omega_0$(n odd)	Infinite rejection frequencies
	($n=1$)	Design frequency

ment values. The prototype is then directly transformed to a transmission-line equivalent, which is further modified to make it suitable for construction in strip line.

The following example illustrates the method. Here the second, third, and fourth harmonics will be completely suppressed theoretically (Fig. 2). A four-reactive element prototype ($n=4$) will be used. (Although three elements would also suffice for infinite attenuation of three harmonic frequencies, the minimum attenuation between harmonics would fall to a lower value.)

DESIGN METHOD

A. Findings the Prototype Circuit Element Values

The prototype filter ($n=4$) is shown in Fig. 3, and tables of element values, including values of A_s, A_p, ω_2' and ω_s' for this circuit as defined in Fig. 1 are given on pp. 37 to 56 of [8], but in reverse order of that in Fig. 3. ([8] is a book published by Telefunken of Western Germany, which contains tables of element values for low-pass filters with equi-ripple attenuation characteristics in both the pass band and stop band. The fineness in the variation of the parameters which distinguish one filter design from its nearest neighbors in the tables, is the quality which lends itself to the method of this paper and permits the bypassing of what might otherwise be quite complicated mathematics.)

Each table gives many designs of the same circuit, all with the same maximum pass-band reflection coefficient (which is different for each table) but with different positions of the finite poles of attenuation. However, only one design of each table fulfills the requirements of this procedure. The immediate problem is to find that set of designs. The tables are so arranged that to do so it is only necessary to establish a simple relationship between ω_2' and ω_s'. One then determines which

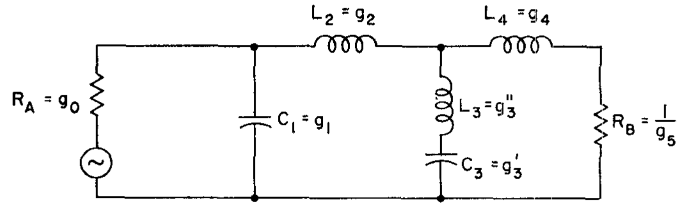

Fig. 3—Low-pass prototype ($n=4$) with element values g_i defined in terms of L (henries), C (farads) and R (ohms).

line of any table (each line specifies a complete prototype filter) satisfies this relationship; the same line of all tables (for this circuit) will also satisfy that relationship. (Lines in the table are indexed by the modular angle θ for easy identification.) This also means, as can be ascertained by examining the tables of element values, that ω_2' and ω_s' will be the same for all prototype circuits in the set, which is a welcome simplification. However, each circuit will have a different pass-band tolerance. We first note the inverse relationship between pass-band and stop-band frequencies which is characteristic of Chebyshev rational function filters [9]

$$\omega_2'\omega_v' = \omega_s'\omega_1'. \tag{1}$$

Since $\omega_1' = 1$ for the prototype, we obtain

$$\omega_v' = \frac{\omega_s'}{\omega_2'}. \tag{2}$$

Next we find ω_v' from the mapping function of Table I. Thus

$$\omega_v^2 = a\omega_2' \tan(\pi\omega_v/2\omega_0) \tag{3}$$

where ω_2' is the frequency of infinite attenuation in the prototype frequency plane. (Recall that we wish to map ω_2' into the second harmonic frequency in the frequency plane of the transmission-line filter.[2]) The bandwidth factor (here it is more like a distortion factor), is defined by

$$a = \operatorname{ctn}(\pi\omega_2/2\omega_0). \tag{4}$$

Next, after substituting $\omega_0 = (3/2)\omega_2$ in (4) we solve for a obtaining $a = 1/\sqrt{3}$, thereby fully specifying the mapping parameters.

Now we substitute this value of a into (3) and solve for ω_v' where $\omega = \omega_0/3$, obtaining

$$\omega_v' = \frac{\omega_2'}{3}. \tag{5}$$

Finally, combining (2) and (5) yield the desired relationship[3]

$$\omega_2'^2 = 3\omega_s'. \tag{6}$$

[2] See Table I for a list of corresponding frequencies.

[3] One might also establish a similar relationship between L_3 and C_3 of Fig. 3, since these elements uniquely determine ω_2'. For this case one would obtain $3\Lambda^2 L_3 C_3 = 1$, whence, with $\Lambda = a\omega_2' = 1.6011$ the relationship $L_3C_3 = 0.129929$ would have to be satisfied. The example to be discussed in this paper (case of $\theta = 27°$ of the element tables for $n=4$) satisfies this relationship with 0.1 per cent error.

The set of suitable prototype circuits is found by substituting into (6) pairs of values ω_2' and ω_s' from one of the tables of element values [8] until agreement is reached between the right and left members, interpolating if necessary. It is thus found that the pair of values $\omega_s'=2.539$, $\omega_2'=2.773$, for modular angle $\theta=27°$, is the best solution since it satisfies (6) with only a 0.1 per cent error. By comparison, the next best pair (with $\theta=28°$), although differing from the above pair by approximately only one per cent, when inserted in (6) yields greater than a one per cent error. The pair of values of ω_s' and ω_2' for $\theta=26°$, again different by only one per cent from those for $\theta=27°$, yields more than a 5 per cent error. Therefore interpolation in this case is probably not necessary and each line indexed by $\theta=27°$ with $\omega_s'=2.539$ and $\omega_2'=2.773$, of the element value tables for the circuit of Fig. 3, may be used for a multiharmonic filter prototype.

B. *Applying the Exact Design Method*

The design formulas in Table II of [7] for the case $n=4$ are here adapted to obtain design formulas for the quarter-wave stubs and connecting lines in Fig. 4. Note that the prototype in Fig. 3 has one two-element branch (L_3, C_3) while the previous formulas [6] were based on the use of a simple shunt-L series-C prototype network. For this reason the formulas for $n=4$ must be revised slightly. The formula for one of the simple shunt stubs must be replaced by two formulas, one for a short-circuited stub Z_3'', and one for an open-circuited stub Z_3', as shown in Fig. 4. The only stub formula in the original design equations that is suitable for replacement is the formula for Z_3, since only that formula owes its derivation to a direct mapping of the corresponding prototype element, without the application of Kuroda's identity [5], [6]. The same is true for one inner stub in most of the formulas for other values of n, and the stub in question can be identified by the fact that its formula has one term only [6]. The set of modified formulas for $n=4$ is given in Table II. In the design formulas of Table II the elements R, L, and C are replaced by g_i, defined in Fig. 3. Next, we choose one of the many usable networks. The particular network chosen here is the first listed for $n=4$ with equal terminations (the case of $\theta=27°$ on p. 37 in [8]). The reason for this choice is that the two extreme values (g_0 and g_3'') have the smallest ratio for any design of the set ($\theta=27°$), and this is helpful in obtaining realistic values of impedance for lines and stubs. The pass-band tolerance, incidentally, is also the least. (The maximum reflection coefficient for this set of element values is one per cent, and the corresponding attenuation loss is 0.004343 db.) The element values are $g_0=1$, $g_1=0.4138$, $g_2=0.7926$, $g_3'=0.6285$, $g_3''=0.2069$, $g_4=0.2498$, $g_5=1$.

Now we compute $\Lambda=a\omega_2'=1.6011$, and then, with the aid of Table II we compute the stub and line impedances as defined in Fig. 4. These impedances (not all of which are final) are $Z_1=3.5093$, $Z_2=0.88658$, $Z_3'=0.33127$, $Z_3''=0.99393$, $Z_4=3.5003$ and $Z_{12}=1.3985$, $Z_{23}=1.8705$, $Z_{34}=1.4000$.

The final step is to convert the two-wire filter of Fig. 4 into a type suitable for strip-line construction, as in Fig. 5. The difference between these two forms of the same filter lies in the third stub (or stubs). In Fig. 5, the third stub, which is open-circuited like the others in that network, consists of two $\lambda_0/4$ sections in tandem in place of two stubs in series [10]. No difficult series-parallel connection is required here, as it is in the filter of Fig. 4.[4] The formulas for converting the series stubs to the tandem stubs are given in Fig. 6, and their proof is given in the Appendix. The stub impedances (Fig. 5) are found to be $Z_{3A}=1.3249$ and $Z_{3B}=3.9747$, thus completing the design in terms of normalized impedances. The computed filter response in Fig. 7 is seen to precisely confirm the foregoing theory.

It is worth noting that the length of the short stubs alone determines the center frequency of infinite attenuation ($3\omega_v$). The relative positions of the other two frequencies of infinite attenuation ($2\omega_v$ and $4\omega_v$) depend on the relation between the two sections of the long stub

$$\frac{Z_{3A}}{Z_{3B}} = \operatorname{ctn}^2 \phi_2$$

where ϕ_2 is the electrical length of each section of the long stub at frequency ω_2 (here $\operatorname{ctn}^2\phi_2=1/3$). Although the element values of the prototype circuit do not in either case determine the conditions for suppression of specific harmonics, they do play a part in determining all other aspects of the filter response.

It is often desirable to know the positions of the virtual open- and short-circuit planes as, for example, when a filter of this type is used with a varactor harmonic generator. The open- and short-circuit planes at the three harmonic frequencies are easily determined as follows: First, in the case of the center (third) harmonic frequency it is clear that the open-circuited quarter-wave long stubs at each end of the filter produce an effective short circuit at both terminals. Second,

[4] The series-parallel stubs Z_3' and Z_3'' of Fig. 4 are not, however, impossible to construct in stripline. They can be made of either

1) an open-circuited shunt stub within an open hollow cylinder which is sandwiched between and short circuited to the ground planes (on the cylinder's far end), or

2) a shunt stub surrounded by a hollow cylinder which is open-circuited to the line and ground planes, but is internally short-circuited to, and supported by, the inner stub on its far end. Interestingly enough, such re-entrant coaxial structures can also be used to reduce by half the over-all length of the tandem double stub Z_3 of Fig. 5. In this case we would have

3) a hollow shunt stub open-circuited on its far end, within which lies on open-circuited stub supported on its far end by the ground planes.

Finally, the tandem double stub can be replaced by either of two electrical equivalents consisting of parallel-coupled lines shown in Figs. 7 and 13 of [6], *i.e.*, the parallel-coupled line resonator and the spur-line resonator, respectively, used as open-circuit stubs.

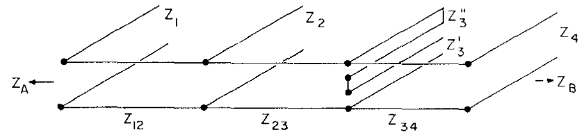

Fig. 4—Balanced transmission-line type multiharmonic rejection filter, derived from the prototype filter of Fig. 3 by the exact design formulas of Table II. (NOTE: All stubs and connecting lines are $\lambda_0/4$ long, where $\omega_0=3\omega_v$).

TABLE II

DESIGN FORMULAS ($n=4$) FOR MULTIHARMONIC REJECTION FILTER

$$Z_1 = Z_A\left(2+\frac{1}{\Lambda g_0 g_1}\right) \qquad Z_{12} = Z_A\left(\frac{1+2\Lambda g_0 g_1}{1+\Lambda g_0 g_1}\right)$$

$$Z_2 = Z_A\left(\frac{1}{1+\Lambda g_0 g_1}+\frac{g_0}{\Lambda g_2(1+\Lambda g_0 g_1)^2}\right) \qquad Z_{23} = \frac{Z_A}{g_0}\left(\Lambda g_2+\frac{g_0}{1+\Lambda g_0 g_1}\right)$$

$$Z_3' = \frac{Z_A}{\Lambda g_0 g_3'} \qquad Z_3'' = Z_A \Lambda g_0 g_3'' \qquad Z_{34} = \frac{Z_A}{g_0 g_5}(1+\Lambda g_4 g_5)$$

$$Z_4 = \frac{Z_A}{g_0 g_5}\left(1+\frac{1}{\Lambda g_4 g_5}\right)$$

Terms are defined in Table I and in Figs. 3 and 4.
These design formulas apply to Fig. 4.
NOTE: To convert the network of Fig. 4 to the strip-line filter of Fig. 5, use the identity defined in Fig. 6 for the double stub (Stub No. 3).

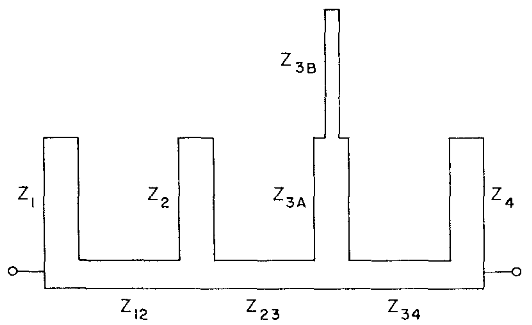

Fig. 5—Exact strip line equivalent of filter of Fig. 4 with easily realizable junctions and equal-length stubs and connecting line sections.

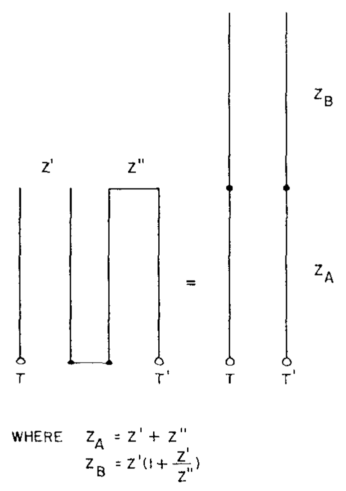

Fig. 6—Two stubs connected in series and the equivalent arrangement of two stubs connected in tandem, with conditions on the line impedances for congruency of terminal reactances.

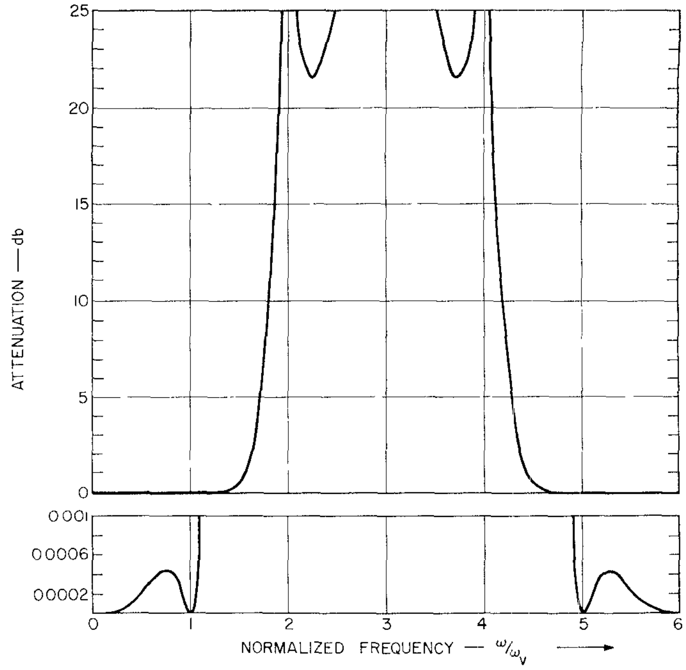

Fig. 7—Computed response of design-example filter as shown in Fig. 5 and described in Part B of text.

at both the lower and upper harmonic frequencies, the long stub places a short circuit at its junction with the main line. This short circuit can now be referred to external positions on the input lines. Using normalized units throughout, we compute the input susceptances at each end by transmission line theory. On the right of the short circuit we have an open-circuited stub whose impedance is 3.5003 in shunt with a short-circuited stub (the connecting line) whose impedance is 1.400. The input susceptance at that end is

$$B_{\text{in}} = \frac{1}{3.5003}\tan\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right) - \frac{1}{1.4000}\cot\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right).$$

At the second harmonic frequency $\omega=2\omega_0/3$, where $B_{\text{in}}=0.495-0.413=0.082$, which also is the susceptance of an open-circuited line of 4.7 electrical degrees. Thus at the second harmonic frequency there is a short circuit plane external to the filter $(90-4.7)=85.3$ degrees from the right-hand terminal. Similarly at the fourth harmonic frequency there is an external open circuit plane 4.7 degrees from the right-hand terminal. Finally, through computations requiring two steps rather than one, the virtual short- and open-circuit positions on the left input line external to the filter are found to be as follows: 1) an open circuit at the second harmonic 25.3 degrees from the terminal, and 2) a short circuit at the fourth harmonic 64.7 degrees from the terminal. In the above calculations all electrical angles are given for specific (harmonic) frequencies which must be used in converting the electrical distances to physical lengths or to units of λ_0, the wavelength at the third harmonic or design frequency.

Appendix

Proof of the Equality of the Networks in Fig. 6

The impedance at Terminals TT' of the left-hand network of Fig. 6 is

$$Z_{\text{in}} = jZ'' \tan \phi - jZ' \operatorname{ctn} \phi \tag{8}$$

where ϕ is the electrical length of each section of transmission line. The impedance of the right-hand network at TT' is the input impedance of a line Z_A, terminated by an impedance $(-jZ_B \operatorname{ctn} \phi)$

$$Z_{\text{in}} = Z_A \frac{-jZ_B \operatorname{ctn} \phi + jZ_A \tan \phi}{Z_A + j(-jZ_B \operatorname{ctn} \phi) \tan \phi} \cdot$$

The above equation is then put in the form of (8):

$$Z_{\text{in}} = j \frac{Z_A^2}{Z_A + Z_B} \tan \phi - \frac{Z_A Z_B}{Z_A + Z_B} \operatorname{ctn} \phi. \tag{9}$$

The Z_{in} of (8) and (9) can be made equal by equating the coefficients of the corresponding terms of their right-hand members,

$$Z'' = \frac{Z_A^2}{Z_A + Z_B} \qquad Z' = \frac{Z_A Z_B}{Z_A + Z_B} \cdot \tag{10}$$

The solution of (10) for Z_A and Z_B yields,

$$Z_A = Z' + Z'' \qquad Z_B = Z(1 + Z'/Z'') \tag{11}$$

which are the formulas of Fig. 6.

Acknowledgment

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